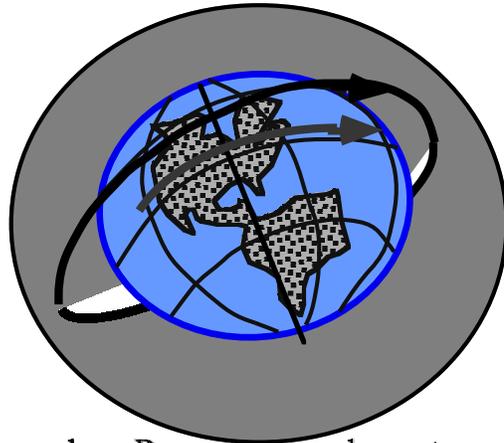


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# SS3011 Space Technology and Applications

## “Orbitology” (cont'd)

Orbital Velocity  
“Orbitology” Orbital Velocity

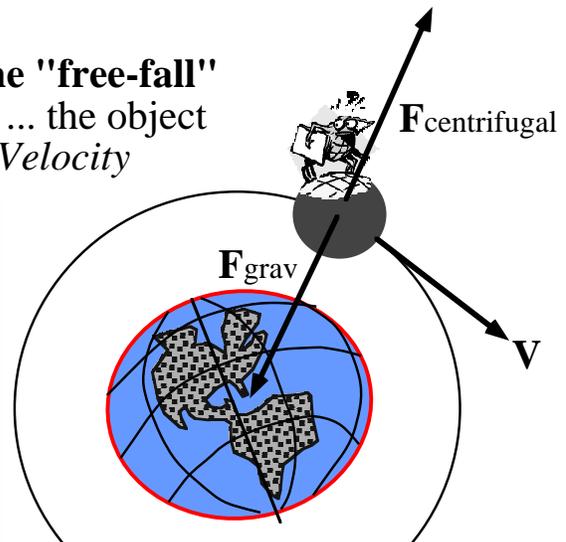


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## Orbital Velocity

- **Object in orbit is actually in "free-fall"**  
*that is ... the object is literally falling around the Earth (or Planet)*
- **When the Centrifugal Force of the "free-fall"** counters the Gravitational Force ... the object is said to have achieved *Orbital Velocity*



Ignoring Drag ... for a Circular orbit

$$\bar{F}_{grav} = \bar{F}_{centrifugal}$$

$$\frac{GMm}{|r|^2} = m \omega^2 |r| = m \left[ \frac{V}{|r|} \right]^2 |r|$$

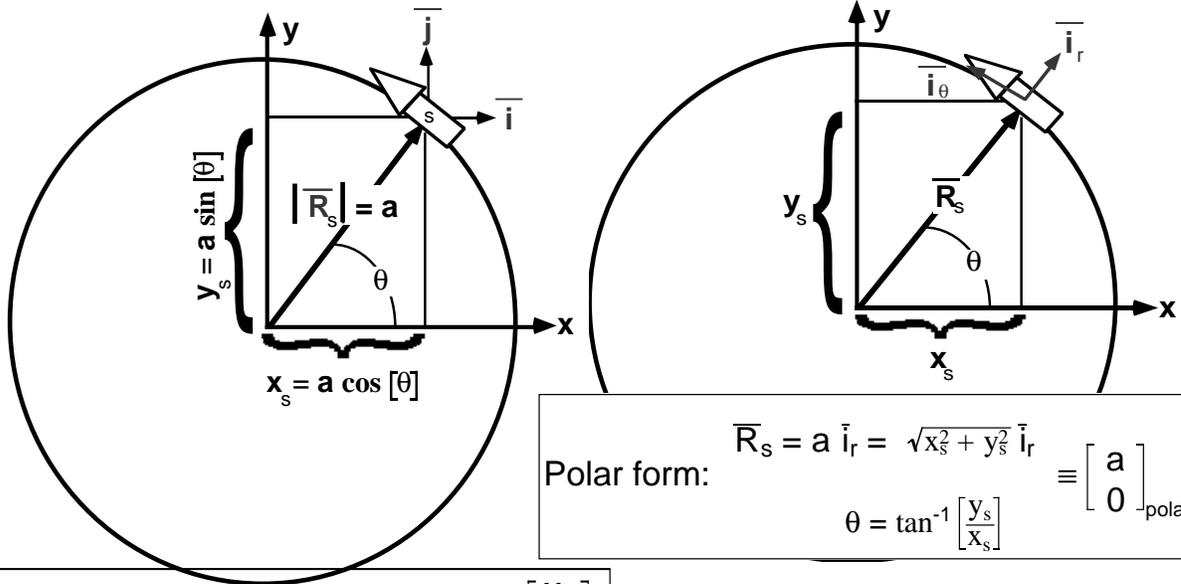
$$V = \sqrt{\frac{GM}{|r|}}$$

Well ... this is a bit simple minded  
reality is actually a bit more complex

# The Velocity Vector

## • Let's Start with a Circular Orbit First

Velocity vector (circular orbit)

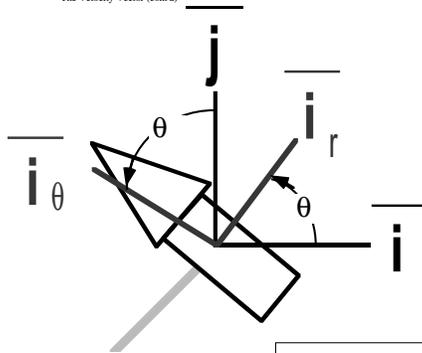


Polar form:  $\bar{R}_s = a \bar{i}_r = \sqrt{x_s^2 + y_s^2} \bar{i}_r \equiv \begin{bmatrix} a \\ 0 \end{bmatrix}_{\text{pola}}$   
 $\theta = \tan^{-1} \left[ \frac{y_s}{x_s} \right]$

Position vector:  $\bar{R}_s = x_s \bar{i} + y_s \bar{j} \equiv \begin{bmatrix} x_s \\ y_s \end{bmatrix}$  Naval Postgraduate School  
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# The Velocity Vector (cont'd)

The Velocity Vector (cont'd)



Transform  $\Rightarrow$  polar  $\uparrow$  cartesian

$$\bar{i} = \bar{i}_r \cos[\theta] - \bar{i}_\theta \sin[\theta]$$

$$\bar{j} = \bar{i}_r \sin[\theta] + \bar{i}_\theta \cos[\theta]$$

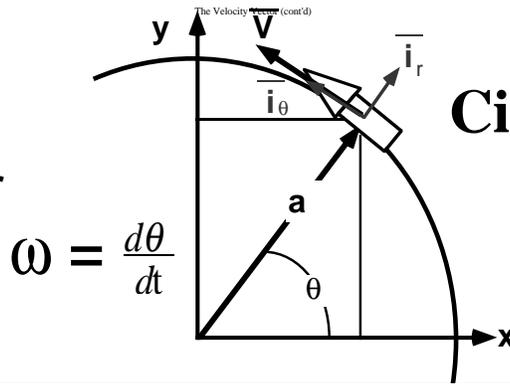
Transform  $\Rightarrow$  cartesian  $\uparrow$  polar

$$\bar{i}_r = \bar{i} \cos[\theta] + \bar{j} \sin[\theta]$$

$$\bar{i}_\theta = -\bar{i} \sin[\theta] + \bar{j} \cos[\theta]$$

# The Velocity Vector (cont'd)

•Velocity vector is time derivative of position vector



## Circular Orbit

$$\omega = \frac{d\theta}{dt}$$

$$\begin{aligned} \bar{V}_s &= \frac{d}{dt}[\bar{R}_s] = \frac{d}{dt}[a \bar{i}_r] = \frac{d}{dt}[a] \bar{i}_r + a \frac{d}{dt}[\bar{i}_r] = \\ &a \frac{d}{dt}[\bar{i} \cos[\theta] + \bar{j} \sin[\theta]] = a [-\bar{i} \sin[\theta] + \bar{j} \cos[\theta]] \frac{d\theta}{dt} = \\ &a \bar{i}_\theta \frac{d\theta}{dt} \equiv a \omega \bar{i}_\theta \end{aligned}$$

# The Velocity Vector (cont'd)

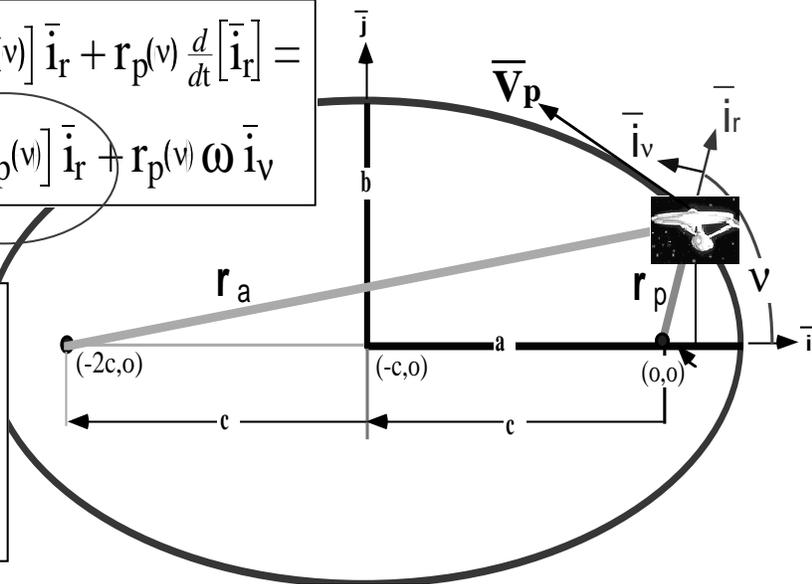
## Elliptical Orbit

$$\bar{V}_p = \frac{d}{dt} \bar{r}_p = \frac{d}{dt} [r_p(v)] \bar{i}_r + r_p(v) \frac{d}{dt} [\bar{i}_r] =$$

$$\frac{d}{dt} [r_p(v)] \bar{i}_r + r_p(v) \omega \bar{i}_v$$

non zero!~

Big deal,  
how do i do  
something  
useful with that?!



# Elliptical Orbit Velocity Vector

Elliptical Orbit Velocity Vector

$$\bar{V}_p = \frac{d}{dt} \bar{r}_p = \frac{d}{dt} [r_p(v)] \bar{i}_r + r_p(v) \omega \bar{i}_v$$

$$\bar{r}_p = \frac{a [1 - e^2]}{1 + e \cos(v)}$$

$$\frac{d}{dt} [r_p(v)] = \frac{d}{dt} \left[ \frac{a [1 - e^2]}{1 + e \cos(v)} \right] = \frac{-a [1 - e^2]}{[1 + e \cos(v)]^2} [-e \sin(v)] \frac{dv}{dt} =$$

$$\frac{a [1 - e^2]}{[1 + e \cos(v)]} \frac{[e \sin(v)]}{[1 + e \cos(v)]} \omega = r_p(v) \omega \frac{[e \sin(v)]}{[1 + e \cos(v)]}$$

# Elliptical Orbit Velocity Vector

Elliptical Orbit Velocity Vector  
(concluded)

(concluded)

$$\bar{V}_p = \frac{d}{dt} \bar{r}_p = \frac{d}{dt} [r_p(v)] \bar{i}_r + r_p(v) \omega \bar{i}_v$$

$$\frac{d}{dt} [r_p(v)] = r_p(v) \omega \frac{[e \sin(v)]}{[1 + e \cos(v)]}$$

$$\bar{V}_p = r_p(v) \omega \left[ \frac{[e \sin(v)]}{[1 + e \cos(v)]} \bar{i}_r + \bar{i}_v \right]$$

## Orbital Speed -- Magnitude of the Velocity Vector (cont'd)

- Taking the Magnitude of the Velocity Vector to get Orbital Speed

$$\bar{V}_p = r_{p(v)} \omega \left[ \frac{[e \sin(v)]}{[1 + e \cos(v)]} \bar{i}_r + \bar{i}_v \right]$$

$$|\bar{V}_p|^2 = [r_{p(v)} \omega]^2 \left[ \left[ \frac{[e \sin(v)]}{[1 + e \cos(v)]} \right]^2 + 1 \right]$$

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## Orbital Speed -- Magnitude of the Velocity Vector (cont'd)

- But from Kepler's second law

$$r^2 \omega = \frac{2 [a^2 \pi \sqrt{1 - e^2}]}{T} \equiv 1$$



$$[r_{p(v)} \omega]^2 = \left[ r_{p(v)} \omega \frac{1}{r_{p(v)}} \right]^2 = \frac{[r_{p(v)}^2 \omega]^2}{r_{p(v)}^2} = \left[ \frac{1}{r_{p(v)}} \right]^2$$

$$|\bar{V}_p|^2 = \left[ \frac{1}{r_{p(v)}} \right]^2 \left[ \left[ \frac{[e \sin(v)]}{[1 + e \cos(v)]} \right]^2 + 1 \right]$$

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# Orbital Speed -- Magnitude of the Velocity Vector (concluded)

- Expanding, Collecting, Simplifying, and *just plain work* results in ....

$$|\bar{V}_p|^2 = \left[ \frac{1}{r_p} \right]^2 \frac{r_p}{a[1-e^2]} \left[ 2 - \frac{r}{a} \right] = \left[ \frac{1^2}{r_p^2} \right] \frac{r_p^2}{a[1-e^2]} \left[ \frac{2}{r} - \frac{1}{a} \right] =$$

$$|\bar{V}_p|^2 = \frac{1^2}{a[1-e^2]} \left[ \frac{2}{r_p} - \frac{1}{a} \right] = \mu \left[ \frac{2}{r_p} - \frac{1}{a} \right]$$

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# Kepler's third law (revisited)

$$\mu \equiv \frac{1^2}{a[1-e^2]} = \frac{\left[ \frac{2 [ a^2 \pi \sqrt{1-e^2} ]}{T} \right]^2}{a[1-e^2]} = \frac{4a^4 \pi^2 [1-e^2]}{a T^2} =$$

$$\frac{4a^4 \pi^2 [1-e^2]}{a T^2} = \boxed{\frac{4a^3 \pi^2}{T^2}}$$

•  $\mu \equiv \frac{1^2}{a[1-e^2]} =$  constant  $\frac{4a^3 \pi^2}{T^2}$

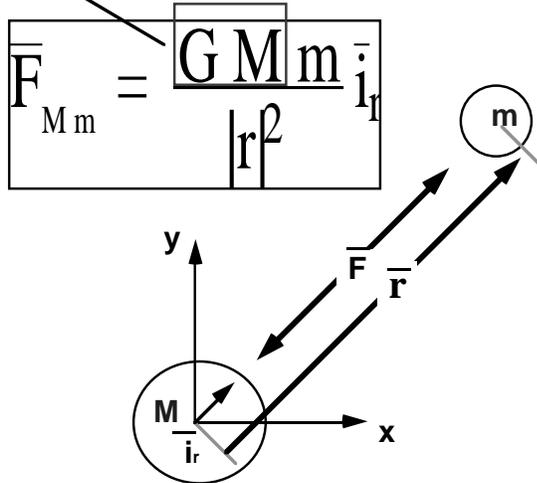
• **Kepler's Third Law: Is not an independent observation ... but instead is a GEOMETRICAL result of Kepler's first and second laws**

# What is $\mu$ ?

What is m?

- In the next lecture we'll show that  $\mu$  is the "planetary gravitational parameter" discussed earlier

$\mu$



Isaac Newton, (1642-1727)  
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## Planetary Gravitational Parameter (revisited)

$$\mu_{\text{earth}} = G M \approx 6.672 \times 10^{-11} \frac{\text{Nt-m}^2}{\text{kg}^2} \times 5.974 \times 10^{24} \text{kg} =$$

$$3.98565 \times 10^{14} \frac{\text{Nt-m}^2}{\text{kg}} = 3.986 \times 10^{14} \frac{\text{m}^3}{\text{sec}^2} = 1.4076 \times 10^{16} \frac{\text{ft}^3}{\text{sec}^2}$$

$$\mu_{\text{moon}} = 4.903 \times 10^3 \frac{\text{m}^3}{\text{sec}^2}$$

$$\mu_{\text{sun}} = 1.327 \times 10^{20} \frac{\text{m}^3}{\text{sec}^2}$$

$$\mu_{\text{Mars}} = 4.269 \times 10^4 \frac{\text{m}^3}{\text{sec}^2}$$

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## Postscript: Magnitude of the Velocity vector

- In the process of demonstrating Kepler's third law, we have also indirectly demonstrated that, for an elliptical orbit

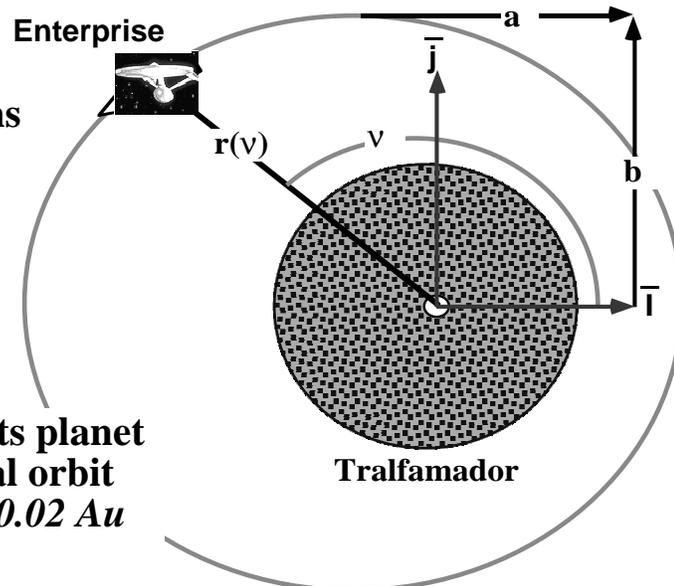
$$|\bar{V}_p|^2 = \mu \left[ \frac{2}{r_p} - \frac{1}{a} \right]$$

- In the next lecture we will show that for an elliptical orbit

$-\frac{\mu}{2a}$	$\frac{ \bar{V}_p ^2}{2}$	$-\frac{\mu}{r_p}$
Total Specific Energy	Specific Kinetic Energy	Specific Potential Energy

## Homework elliptical orbits: 2

- Sellers, Chapter 4, sections 4.1, 4.2, 4.3



- Starship Enterprise orbits planet *Tralfamador* in an elliptical orbit with semi-major axis  $a = 0.02 \text{ Au}$  (\*astronomical units) and semi-minor axis  $b = 0.01 \text{ Au}$

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## Homework elliptical orbits: 2

- Given that Tralfamador has a mass 100 times greater than the Earth

i) Compute the period of the orbit (check your units!)

Hint 1

$$\mu_{\text{earth}} = G M \approx 6.672 \times 10^{-11} \frac{\text{Nt-m}^2}{\text{kg}^2} \times 5.974 \times 10^{24} \text{kg} =$$
$$3.98565 \times 10^{14} \frac{\text{Nt-m}^2}{\text{kg}} = 3.986 \times 10^{14} \frac{\text{m}^3}{\text{sec}^2} = 1.4076 \times 10^{16} \frac{\text{ft}^3}{\text{sec}^2}$$

**\*AU  $\approx$  150,000,000 kilometers**

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## Homework elliptical orbits: 2

- Calculate the Magnitude of the velocity at
  - i) perifamador
  - ii) apfamador

Hint 2

$$\frac{r_{\text{max}} + r_{\text{min}}}{2} = a$$

$$|\bar{V}_p|^2 = \mu \left[ \frac{2}{r_p} - \frac{1}{a} \right]$$

$$\frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}} = e$$

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## Homework elliptical orbits: 2

- Calculate the  $\Delta V$  required to achieve "Escape velocity" at
  - i) perifamador
  - ii) apfamador

Hint 3: i.e. find the change in velocity that puts the satellite on an elliptical trajectory with

$$a = \infty$$

$$V_{\text{escape}}^2 = \frac{\left[ \frac{2\mu}{r} - \frac{\mu}{a} \right]}{\lim_{a \rightarrow \infty}} = \frac{2\mu}{r}$$

$$\text{"}\Delta V\text{"}_{\text{escape}} = V_{\text{escape}} - V_p$$